Division of Strength of Materials and Structures
Faculty of Power and Aeronautical Engineering

# Finite element method (FEM1)

## Lecture 10B. 2D frame element

05.2025

# Examples of frame structures

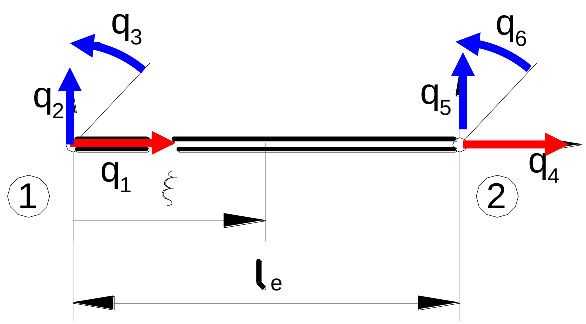
Motor hang glider structure

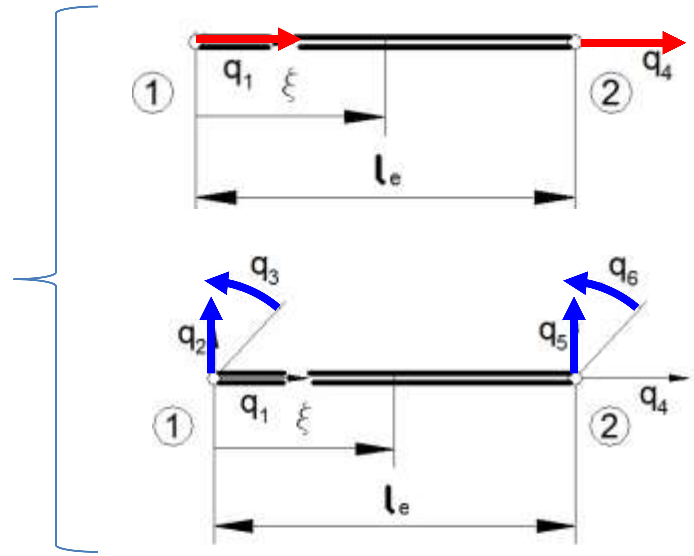
Engine bed

Frame construction of the hall

Bicycle frame

# 2D frame element in local coordinate system – local stiffness matrix



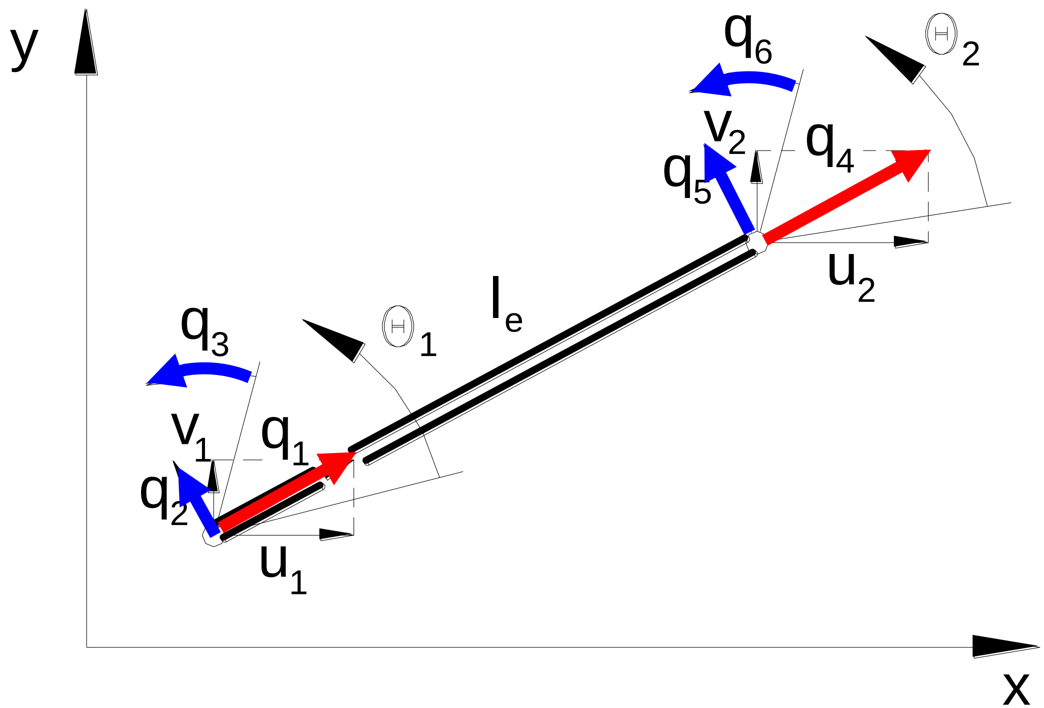




$$[k]_e = \begin{bmatrix} \frac{EA}{l_e} & 0 & 0 & -\frac{EA}{l_e} & 0 & 0 \\ 0 & \frac{12EI}{l_e^3} & \frac{6EI}{l_e^2} & 0 & \frac{-12EI}{l_e^3} & \frac{6EI}{l_e^2} \\ 0 & \frac{6EI}{l_e^2} & \frac{4EI}{l_e} & 0 & \frac{-6EI}{l_e^2} & \frac{2EI}{l_e} \\ \frac{-EA}{l_e} & 0 & 0 & \frac{EA}{l_e} & 0 & 0 \\ 0 & \frac{-12EI}{l_e^3} & \frac{-6EI}{l_e^2} & 0 & \frac{12EI}{l_e^3} & \frac{-6EI}{l_e^2} \\ 0 & \frac{6EI}{l_e^2} & \frac{2EI}{l_e} & 0 & \frac{-6EI}{l_e^2} & \frac{4EI}{l_e} \end{bmatrix}$$

$$[k]_e = \frac{EA}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k]_e = \frac{2EI}{l_e^3}\begin{bmatrix} 6 & 3l_e & -6 & 3l_e \\ 3l_e & 2l_e^2 & -3l_e & l_e^2 \\ -6 & -3l_e & 6 & -3l_e \\ 3l_e & l_e^2 & -3l_e & 2l_e^2 \end{bmatrix}$$

# 2D frame element in global coordinate system – local and global nodal parameters



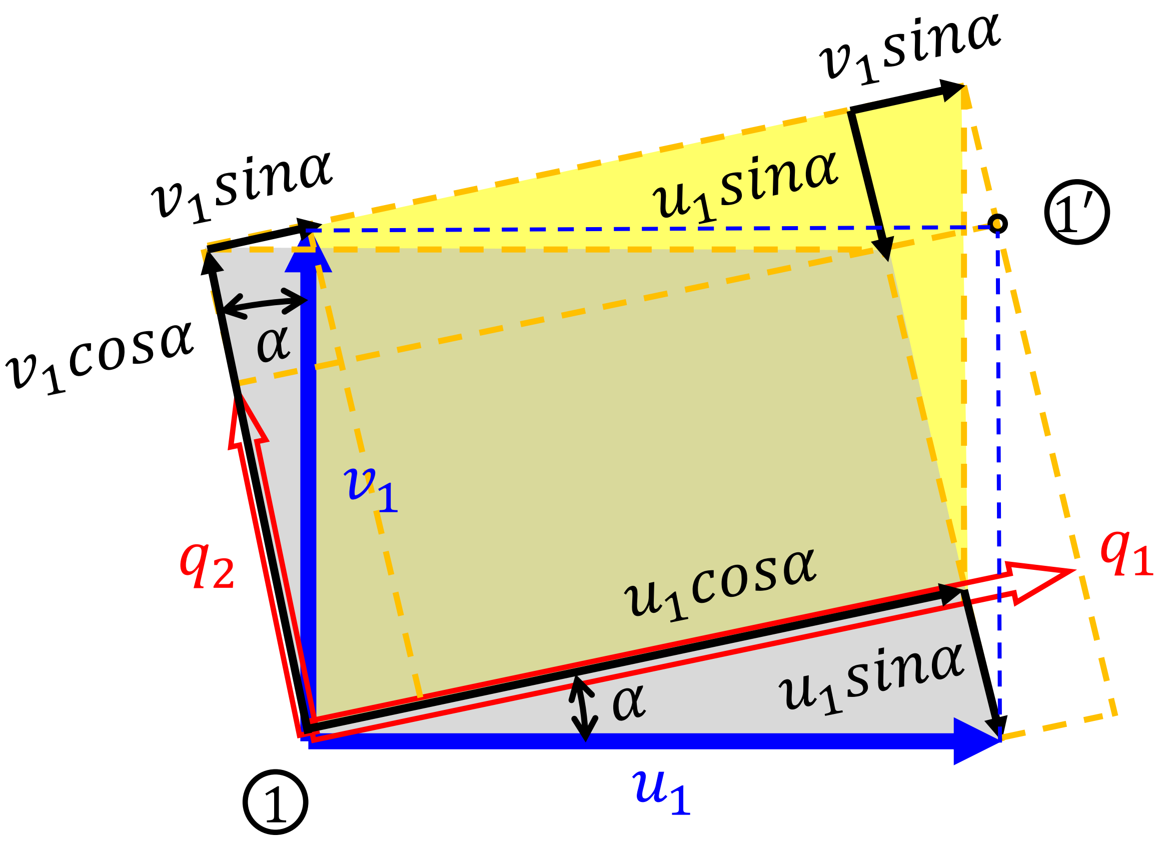

Local vector of nodal parameters:

$$\{q\}_e = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}$$

Global vector of nodal parameters:

$$\{q_g\}_e = \begin{Bmatrix} u_1 \\ v_1 \\ \Theta_1 \\ u_2 \\ v_2 \\ \Theta_2 \end{Bmatrix}$$

Transformation of degrees of freedom from global to local coordinate system

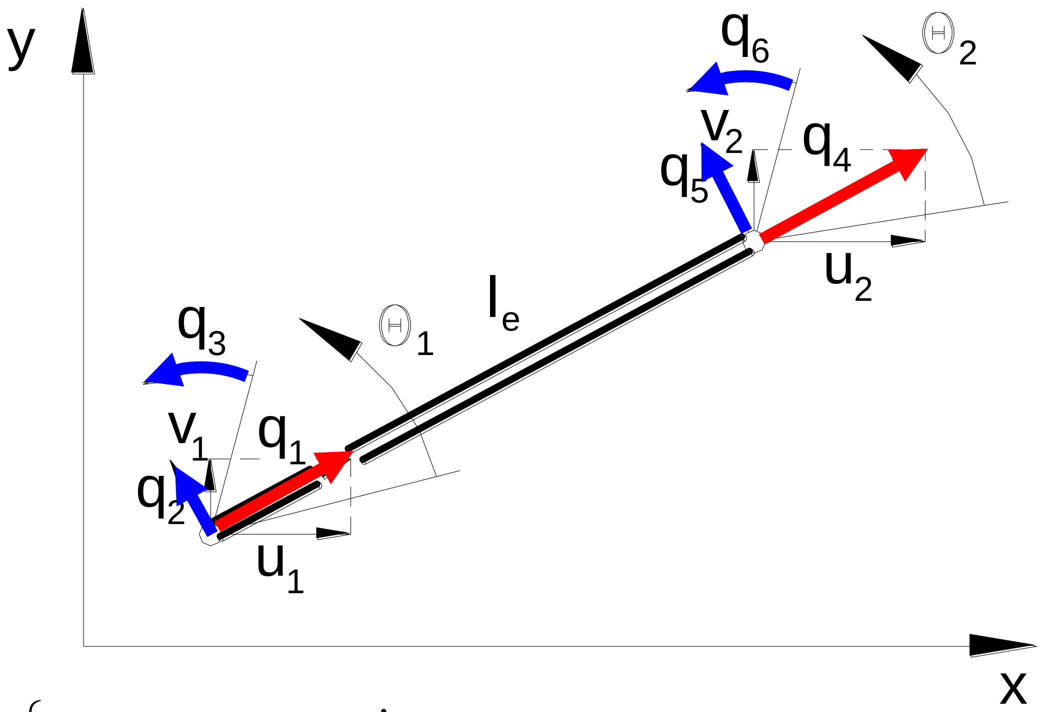



$$q_1 = u_1 \cdot cos\alpha + v_1 \cdot sin\alpha = c \cdot u_1 + s \cdot v_1$$

$$q_2 = -u_1 \cdot sin\alpha + v_1 \cdot cos\alpha = -s \cdot u_1 + c \cdot v_1$$

$$(c = cos\alpha \quad ; \quad s = sin\alpha)$$

# 2D frame element in global coordinate system – transformation matrix




Local vector of nodal parameters:

$$\{q\}_e = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}$$

Global vector of nodal parameters:

$$\{q_g\}_e = \begin{Bmatrix} u_1 \\ v_1 \\ \Theta_1 \\ u_2 \\ v_2 \\ \Theta_2 \end{Bmatrix}$$

$$\begin{cases} q_1 = u_1 \cos\alpha + \upsilon_1 \sin\alpha, \\ q_2 = -u_1 \sin\alpha + \upsilon_1 \cos\alpha, \\ q_3 = \theta_1. \end{cases}$$

$c = cos\alpha$

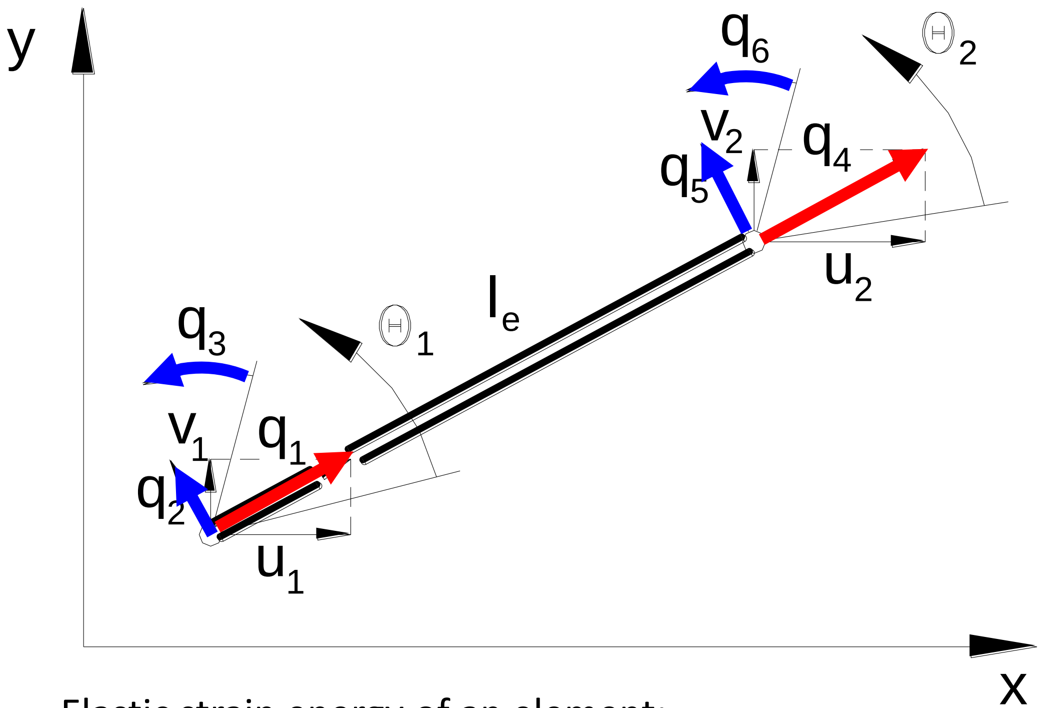
$s = sin\alpha$

Transformation matrix:

$$[T_r] = \begin{bmatrix} c & s & 0 & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}_e = [T_r] \begin{Bmatrix} u_1 \\ \upsilon_1 \\ \theta_1 \\ u_2 \\ \upsilon_2 \\ \theta_2 \end{Bmatrix} = [T_r] \cdot \{q_g\}_e$$

# 2D frame element in global coordinate system




Local vector of nodal parameters:

$$\{q\}_e = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}$$

Global vector of nodal parameters:

$$\{q_g\}_e = \begin{Bmatrix} u_1 \\ v_1 \\ \Theta_1 \\ u_2 \\ v_2 \\ \Theta_2 \end{Bmatrix}$$

Elastic strain energy of an element:

$$U_e = \frac{1}{2} \lfloor q \rfloor_e [k]_e \{q\}_e = \frac{1}{2} \lfloor q_g \rfloor_e [T_r]^T [k]_e [T_r] \{q_g\}_e ,$$

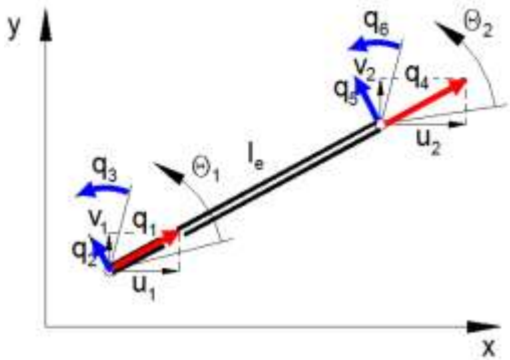
$$U_e = \frac{1}{2} \lfloor q_g \rfloor_e \left[ k^g \right]_e \{q_g\}_e ,$$

Element stiffness matrix:

$$\left[ k^g \right]_e = [T_r]^T [k]_e [T_r]$$

# Global 2D frame element stiffness matrix

$$\left[k^{g}\right]_e = \left[T_r\right]^T \left[k\right]_e \left[T\right]$$




$$[k]_e = \begin{bmatrix} \frac{EA}{l_e} & 0 & 0 & -\frac{EA}{l_e} & 0 & 0 \\ 0 & \frac{12EI}{l_e^3} & \frac{6EI}{l_e^2} & 0 & \frac{-12EI}{l_e^3} & \frac{6EI}{l_e^2} \\ 0 & \frac{6EI}{l_e^2} & \frac{4EI}{l_e} & 0 & \frac{-6EI}{l_e^2} & \frac{2EI}{l_e} \\ \frac{-EA}{l_e} & 0 & 0 & \frac{EA}{l_e} & 0 & 0 \\ 0 & \frac{-12EI}{l_e^3} & \frac{-6EI}{l_e^2} & 0 & \frac{12EI}{l_e^3} & \frac{-6EI}{l_e^2} \\ 0 & \frac{6EI}{l_e^2} & \frac{2EI}{l_e} & 0 & \frac{-6EI}{l_e^2} & \frac{4EI}{l_e} \end{bmatrix}$$

$$[T] = \begin{bmatrix} c & s & 0 & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[T_r]^T = \begin{bmatrix} c & -s & 0 & 0 & 0 & 0 \\ s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & -s & 0 \\ 0 & 0 & 0 & s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$